Asymptotic symmetries in 3d gravity with torsion

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Abstract

We study the nature of asymptotic symmetries in topological 3d gravity with torsion. After introducing the concept of asymptotically anti-de Sitter configuration, we find that the canonical realization of the asymptotic symmetry is characterized by the Virasoro algebra with classical central charge, the value of which is the same as in general relativity: $c = 3\ell/2G$.

I. INTRODUCTION

Three-dimensional (3d) gravity has been used as a theoretical laboratory to test some of the conceptual problems of quantum gravity. The phase space structure of 3d gravity is known to be of great importance not only at the classical level, but also for a clear understanding of the related quantum structure [1,2]. In particular, the Virasoro algebra of the asymptotic symmetry plays a central role in our understanding of the quantum nature of black hole [3–10]. One can observe, however, that the analysis of these issues has been carried out only in *Riemannian* spacetime of general relativity. In the present paper we begin an investigation of the asymptotic structure of 3d gravity in the context of *Riemann–Cartan* geometry, a geometry possessing a metric compatible connection, with both the curvature and the torsion of the underlying spacetime manifold [11,12]. In this way, we expect to clarify the influence of spacetime geometry on the boundary dynamics.

Dynamics of a theory is determined not only by the action, but also by the asymptotic conditions. The dynamical role of asymptotic conditions is best seen in topological theories, where the non-trivial dynamics is bound to exist only at the boundary. General action for topological 3d gravity in Riemann–Cartan spacetime has been proposed by Baekler and Mielke [13,14]. For particular values of parameters, this action leads to the *teleparallel* (Weizenböck) geometry, defined by the requirement of vanishing curvature [15–17,12]. Teleparallel geometry is, in a sense, complementary to Riemannian: curvature vanishes, and torsion remains to characterize the parallel transport. We choose this teleparallel framework to study the asymptotic structure of spacetime in the presence of torsion. After introducing the concept of asymptotically anti-de Sitter (AdS) field configuration and performing the canonical analysis, we find that the teleparallel spacetime has the same asymptotic structure as the Riemannian spacetime in general relativity.

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The paper is organized as follows. In Sect. II we introduce Riemann–Cartan spacetime as a general geometric arena for 3d gravity with torsion, and show that there is a specific choice of parameters which leads to the teleparallel description of gravity. In Sect. III we construct two exact solution of the resulting teleparallel theory: the AdS solution and the black hole. Then, in Sect. IV, we introduce the concept of asymptotically AdS configuration. The symmetry of such a configuration, the asymptotic symmetry, is shown to be the same as in general relativity — the conformal symmetry. In the next two sections, the gauge structure of the theory is incorporated into the canonical formalism. First, the general Hamiltonian structure of the theory is derived in Sect. V. After that, in Sect. VI, we construct the canonical gauge generators compatible with the adopted asymptotic conditions. The construction is realized with the help of appropriate boundary terms [18], which are interpreted as the conserved charges of the theory. The investigation of the Poisson bracket algebra of the asymptotic generators leads to the central result of the paper: the asymptotic symmetry is characterized by the classical Virasoro algebra with central charge, the value of which is the same as in Riemannian spacetime of general relativity: $c = 3\ell/2G$ [4–10]. Finally, section VII is devoted to concluding remarks, while Appendices contain some technical details.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the first letters of both alphabets $(a, b, c, ...; \alpha, \beta, \gamma, ...)$ run over 1,2, the middle alphabet letters $(i, j, k, ...; \mu, \nu, \lambda, ...)$ run over 0,1,2; the tetrad field $b^i{}_\mu$ and its dual $h_i{}^\mu$ are used to convert Greek and Latin indices into each other; $\eta_{ij} = (+, -, -)$ and $g_{\mu\nu} = b^i{}_\mu b^j{}_\nu \eta_{ij}$ are the metric components in the tangent and coordinate frame; totally antisymmetric tensor ε^{ijk} and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized so that $\varepsilon^{012} = 1$.

II. BASIC DYNAMICAL FEATURES

1. Three-dimensional gravity with torsion can be formulated as Poincaré gauge theory, with an underlying geometric structure described by *Riemann–Cartan* space U_3. Basic gravitational variables of the theory are the triad field $b^i{}_\mu$ and the Lorentz connection $A^{ij}{}_\mu$. Their gauge transformations are local Lorentz rotations and local translations, with the parameters ε^{ij} and ξ^μ, respectively [11,12]:

$$
\begin{aligned}
\delta_0 b^i{}_\mu &= \varepsilon^i{}_k b^k{}_\mu - \xi^\rho{}_{,\mu} b^i{}_\rho - \xi^\rho \partial_\rho b^i{}_\mu \,, \\
\delta_0 A^{ij}{}_\mu &= -\nabla_\mu \varepsilon^{ij} - \xi^\rho{}_{,\mu} A^{ij}{}_\rho - \xi^\rho \partial_\rho A^{ij}{}_\mu \,.
\end{aligned}
$$

The related field strengths $T^i{}_{\mu\nu}$ and $R^{ij}{}_{\mu\nu}$ are geometrically identified with the torsion and the curvature:

$$
\begin{aligned}
T^i{}_{\mu\nu} &= \partial_\mu b^i{}_\nu + A^i{}_{m\mu} b^m{}_\nu - (\mu \leftrightarrow \nu) \,, \\
R^{ij}{}_{\mu\nu} &= \partial_\mu A^{ij}{}_\nu + A^i{}_{m\mu} A^{mj}{}_\nu - (\mu \leftrightarrow \nu) \,.
\end{aligned}
$$

Note that, in this approach, metric is not an independent field variable: $g_{\mu\nu}$ is defined in terms of $b^i{}_\mu$ and the tangent space metric η by the relation $g_{\mu\nu} = b^i{}_\mu b^i{}_\nu \eta_{ij}$.

In $d = 3$, we can simplify the notation by introducing

$$\omega_{i\mu} = -\tfrac{1}{2}\varepsilon_{ijk}A^{jk}{}_{\mu}\,, \qquad \theta_i = -\tfrac{1}{2}\varepsilon_{ijk}\varepsilon^{jk}\,,$$
$$R_{i\mu\nu} = -\tfrac{1}{2}\varepsilon_{ijk}R^{jk}{}_{\mu\nu}\,.$$

Then, the transformation laws of the gauge fields take the form

$$\begin{aligned}\delta_0 b^i{}_\mu &= \varepsilon^{ijk}\theta_j b_{k\mu} - \xi^\rho{}_{,\mu}b^i{}_\rho - \xi^\rho\partial_\rho b^i{}_\mu\\ \delta_0\omega^i{}_\mu &= -\nabla_\mu\theta^i - \xi^\rho{}_{,\mu}\omega^i{}_\rho - \xi^\rho\partial_\rho\omega^i{}_\mu\,,\end{aligned} \tag{2.1a}$$

where $\nabla_\mu\theta^i = \partial_\mu\theta^i + \varepsilon_{imn}\omega^m{}_\mu\theta^n$, and the field strengths are given as

$$\begin{aligned}T^i{}_{\mu\nu} &= \partial_\mu b^i{}_\nu + \varepsilon^{ijk}\omega_{j\mu}b_{k\nu} - (\mu\leftrightarrow\nu)\,,\\ R_{i\mu\nu} &= \partial_\mu\omega_{i\nu} - \partial_\nu\omega_{i\mu} + \varepsilon_{ijk}\omega^j{}_\mu\omega^k{}_\nu\,.\end{aligned} \tag{2.1b}$$

2. Dynamical structure of the theory is determined by an action integral (the important role of boundary conditions will be discussed later). Direct generalization of Einstein's theory to the U_3 space leads to Einstein–Cartan theory:

$$I_{EC} = -a\int d^3x bR\,, \qquad a = \frac{1}{16\pi G}\,.$$

Staying in the realm of Riemannian geometry (vanishing torsion), Witten [2] demonstrated that Einstein–Cartan theory with cosmological constant is equivalent to the standard gauge theory of Chern–Simons type. An interesting extension of these ideas to Riemann–Cartan space (non-vanishing torsion) has been proposed by Baekler and Mielke [13,14]. They studied an action constructed out of the following topological or topological-like terms:

$$\begin{aligned}I_1 &= -\int d^3x b(aR+2\Lambda) = \int d^3x\varepsilon^{\mu\nu\rho}\left(ab^i{}_\mu R_{i\nu\rho} - \tfrac{1}{3}\Lambda\varepsilon_{ijk}b^i{}_\mu b^j{}_\nu b^k{}_\rho\right)\,,\\ I_2 &= \int d^3x\varepsilon^{\mu\nu\rho}\left(\omega^i{}_\mu\partial_\nu\omega_{i\rho} + \tfrac{1}{3}\varepsilon_{imn}\omega^i{}_\mu\omega^m{}_\nu\omega^n{}_\rho\right)\,,\\ I_3 &= \tfrac{1}{2}\int d^3x\varepsilon^{\mu\nu\rho}b^i{}_\mu T_{i\nu\rho}\,,\end{aligned} \tag{2.2a}$$

where Λ is a constant. The first term describes Einstein–Cartan theory with cosmological constant, the second term is the Chern–Simons action for the Lorentz connection, and the third term represents an action of the translational Chern–Simons type [19]. The general Baekler–Mielke action reads:

$$I = I_1 + \alpha_2 I_2 + \alpha_3 I_3 + I_M\,, \tag{2.2b}$$

where I_M is an action for matter fields.

Varying the action with respect to $b^i{}_\mu$ and $\omega^i{}_\mu$, we obtain the field equations:

$$\begin{aligned}\varepsilon^{\mu\nu\rho}\left[aR_{i\nu\rho} - \Lambda\varepsilon_{ijk}b^j{}_\nu b^k{}_\rho + \alpha_3 T_{i\nu\rho}\right] &= \tau^\mu{}_i\,,\\ \varepsilon^{\mu\nu\rho}\left[aT_{i\nu\rho} + \alpha_2 R_{i\nu\rho} + \alpha_3\varepsilon_{ijk}b^j{}_\nu b^k{}_\rho\right] &= \sigma^\mu{}_i\,,\end{aligned} \tag{2.3}$$

where τ and σ are the energy-momentum and spin tensors of matter fields, respectively. From these equations one can calculate explicitly the torsion and the curvature of the spacetime.

For our purposes, it is sufficient to consider only the solutions of these equations in vacuum, where $\tau = \sigma = 0$. In that case, and for $\alpha_2\alpha_3 - a^2 \neq 0$, we find

$$T_{ijk} = A\varepsilon_{ijk}\,, \qquad R_{ijk} = B\varepsilon_{ijk}\,, \tag{2.4}$$

with

$$A \equiv \frac{\alpha_2\Lambda + \alpha_3 a}{\alpha_2\alpha_3 - a^2}\,, \qquad B \equiv -\frac{(\alpha_3)^2 + a\Lambda}{\alpha_2\alpha_3 - a^2}\,.$$

In Riemann–Cartan space U_3, the Lorentz connection can be expressed in terms of the Levi–Civita connection Δ and the contortion K as $A = \Delta + K$ [11,12]. Substituting this expression into the definition of the curvature tensor $R^{ij}{}_{\mu\nu}(A)$ leads to the geometric identity

$$R^{ij}{}_{\mu\nu}(A) = R^{ij}{}_{\mu\nu}(\Delta) + \left[\nabla_\mu K^{ij}{}_\nu - K^i{}_{s\mu}K^{sj}{}_\nu - (\mu \leftrightarrow \nu)\right]\,.$$

Then, by combining this identity with the vacuum field equations (2.4), we obtain the following expression for the Riemannian piece of the U_3 curvature:

$$R^{ij}{}_{\mu\nu}(\Delta) = -\Lambda_{\text{eff}}(b^i{}_\mu b^j{}_\nu - b^i{}_\nu b^j{}_\mu)\,, \qquad \Lambda_{\text{eff}} \equiv B - \frac{1}{4}A^2\,, \tag{2.5}$$

where Λ_{eff} is the effective cosmological constant. Looking at this equation as an equation for the metric, we see that the metric of our spacetime is maximally symmetric; for $\Lambda_{\text{eff}} < 0$ ($\Lambda_{\text{eff}} > 0$) it has the anti-de Sitter (de Sitter) form.

3. At the end of this section, we would like to comment on two special cases of the Baekler–Mielke action.

For $\alpha_2 = \alpha_3 = 0$ (Witten's choice [2]), we have

$$T_{ijk} = 0\,, \qquad R_{ijk} = \frac{\Lambda}{a}\,\varepsilon_{ijk}\,. \tag{2.6}$$

The torsion vanishes, and the geometry of spacetime becomes *Riemannian*.

Another interesting choice is $(\alpha_3)^2 + a\Lambda = 0$. It yields the field equations

$$T_{ijk} = -\frac{\alpha_3}{a}\,\varepsilon_{ijk}\,, \qquad R_{ijk} = 0\,, \tag{2.7a,b}$$

which are "geometrically dual" to those of Witten: the curvature vanishes, and the geometry becomes *teleparallel*.

Having in mind our intention to study the role of torsion in the boundary dynamics, we restrict our attention to the teleparallel case (2.7). Since the field equations are independent of α_2, we also assume $\alpha_2 = 0$. The effective cosmological constant is now negative:

$$\Lambda_{\text{eff}} = -\frac{1}{4}A^2 \equiv -\frac{1}{\ell^2} < 0\,. \tag{2.8a}$$

After introducing the constant ℓ by the relation $A = 2/\ell$, these conditions are summarized as

$$\alpha_2 = 0\,, \qquad \alpha_3 = -\frac{2a}{\ell}\,, \qquad \Lambda = -\frac{4a}{\ell^2}\,, \tag{2.8b}$$

and the general action (2.2b) in the absence of matter reduces to the form

$$I = -a\int d^3x\, b\left(R + \frac{1}{\ell}\,\varepsilon^{ijk}\,T_{ijk} - \frac{8}{\ell^2}\right)\,. \tag{2.9}$$

III. EXACT VACUUM SOLUTIONS

We now direct our attention to the exact classical solutions of the vacuum field equations (2.7). In this regard, it is useful to note that equations (2.7b) and (2.5) are equivalent, provided equation (2.7a) holds. As a consequence, our search for the exact solutions will be based on the following strategy:

i) we shall first find a solution of equation (2.5) for the metric;
ii) given the metric, we shall procede to find a solution for the triad field;
iii) finally, we shall use equation (2.7a) to determine the connection.

After that, equation (2.7b) will be automatically satisfied.

The first step in the above procedure is very simple, since the form of the metric in maximally symmetric 3d spaces is well known [20].

A. Teleparallel AdS solution

As the first solution of (2.5) with $\Lambda_{\text{eff}} = -1/\ell^2$, we display the metric of the AdS solution in static coordinates $x^\mu = (t, r, \varphi)$:

$$ds^2 = f^2 dt^2 - f^{-2} dr^2 - r^2 d\varphi^2 \,, \qquad f^2 \equiv 1 + \frac{r^2}{\ell^2} \,. \tag{3.1}$$

The related triad field can be chosen to have the simple, diagonal form:

$$b^0 = f dt \,, \qquad b^1 = f^{-1} dr \,, \qquad b^2 = r d\varphi \,, \tag{3.2}$$

where $b^i = b^i{}_\mu dx^\mu$. It produces the metric (3.1) via $ds^2 = b^i b^j \eta_{ij}$.

The connection is determined from equation (2.7a). Introducing the light-cone coordinates

$$x^\pm = \frac{1}{\ell} x^0 \pm x^2 \,,$$

the solution for the connection 1-form $\omega^i = \omega^i{}_\mu dx^\mu$ is given as

$$\omega^0 = f dx^- \,, \qquad \omega^1 = \frac{1}{\ell f}\, dr \,, \qquad \omega^2 = -\frac{r}{\ell}\, dx^- \,. \tag{3.3}$$

Thus, equations (3.2) and (3.3) represent the exact AdS vacuum solution of our theory. This solution is defined in the realm of the teleparallel geometry, and should not be confused with the AdS solution (3.1) in Riemannian geometry.

The symmetries of the AdS solution are discussed in Appendix A.

Comment. Given the metric (3.1), the choice of the AdS pair $(b^i{}_\mu, \omega^i{}_\mu)$ is not unique, in the sense that any Lorentz transform of a particular solution yields also a solution of the theory. On the other hand, as a consequence of $R_{i\mu\nu} = 0$, there exists a solution with the trivial connection $\tilde{\omega}^i{}_\mu = 0$. Any other vacuum connection can be written as a Lorentz transform of $\tilde{\omega}^i{}_\mu$. This is especially true for our vacuum connection (3.3):

$$-\varepsilon^{ijk} \omega_{k\mu} = \Lambda^i{}_k \, \Lambda^{jk}{}_{,\mu} \,, \qquad \Lambda^i{}_k \Lambda^j{}_l \, \eta^{kl} = \eta^{ij} \,.$$

From this equation, we find the following particular solution for the Lorentz matrix Λ:

$$\Lambda^i{}_j = \begin{pmatrix} -f & \dfrac{r}{\ell}\sin x^- & -\dfrac{r}{\ell}\cos x^- \\ 0 & -\cos x^- & -\sin x^- \\ \dfrac{r}{\ell} & -f\sin x^- & f\cos x^- \end{pmatrix}.$$

The general solution is obtained by the replacement $\Lambda \to \Lambda\Lambda_c$, where Λ_c is a constant Lorentz matrix.

Now, when we know the Lorentz matrix which transforms the trivial connection $\tilde{\omega}^i_\mu = 0$ into our $\omega^i{}_\mu$, we can easily find the related triad field $\tilde{b}^i{}_\mu$ as $\tilde{b}^i{}_\mu = \Lambda_k{}^i\, b^k{}_\mu$:

$$\tilde{b}^i{}_\mu = \begin{pmatrix} -f^2 & 0 & -\dfrac{r^2}{\ell} \\ -\dfrac{r}{\ell}\, f\sin x^- & -f^{-1}\cos x^- & -rf\sin x^- \\ \dfrac{r}{\ell}\, f\cos x^- & -f^{-1}\sin x^- & rf\cos x^- \end{pmatrix}.$$

Thus, $\tilde{b}^i{}_\mu$ and $\tilde{\omega}^i{}_\mu = 0$ are also vacuum solutions of the field equations (2.7), which differ from those given in (3.2) and (3.3) by the local Lorentz rotation.

Our vacuum triad (3.2) is not well defined at $r = 0$, while $\tilde{b}^i{}_\mu$ is. If we are only interested in the asymptotic region (as we are), both vacuum solutions are acceptable.

B. Teleparallel black hole

Another well known solution of equation (2.5) is the BTZ black hole [21]. In static coordinates (t, r, φ), the black hole is defined by the metric (in units $4G = 1$)

$$\begin{aligned} &ds^2 = N^2dt^2 - N^{-2}dr^2 - r^2(d\varphi + N_\varphi dt)^2\,, \\ &N^2 \equiv \left(-2m + \frac{r^2}{\ell^2} + \frac{J^2}{r^2}\right), \qquad N_\varphi \equiv \frac{J}{r^2}\,, \end{aligned} \tag{3.4}$$

with $0 \le \varphi < 2\pi$. Although the AdS vacuum and the black hole are locally isometric solutions, they are globally distinct: the black hole describes a conic geometry obtained by a geometric identification of points in AdS space [21]. The two parameters m and J are related to the global properties of conic geometries known as "missing angle" and "time jump" [1]. The thorough analysis of Ref. [21] shows that all physically acceptable solutions of our theory are exhausted by the two-parameter black hole solution (3.4). As we shall see later, the parameters m and J have the physical meaning of energy and angular momentum.

The black hole triad and connection are not uniquely defined by the metric (3.4). Although all possible solutions are locally equivalent, they may differ globally. It can be shown, for example, that the solution with everywhere vanishing connection *is not* globally well defined for all the values of m and J.

In what follows, we adopt the simple ansatz for the triad field (see also Ref. [22]):

$$\begin{aligned} &b^0 = N dt\,, \qquad b^1 = N^{-1}dr\,, \\ &b^2 = r(d\varphi + N_\varphi dt)\,. \end{aligned} \tag{3.5}$$

The connection is, again, determined by equation (2.7a), which we rewrite in the form

$$\frac{1}{\ell}\,\varepsilon^i{}_{mn}b^m b^n = db^i - \varepsilon^i{}_{jk}\omega^k b^j\,.$$

Explicit calculation for $i = 0, 1, 2$ yields:

$$\begin{aligned}
&\omega^0{}_1 = 0\,, \qquad \omega^1{}_0 = \omega^1{}_2 = 0\,, \qquad \omega^2{}_1 = 0\,, \\
&\frac{2}{\ell}b^2{}_0 = \frac{b^0{}_0{}'}{b^1{}_1} + \omega^2{}_0 + \frac{\omega^1{}_1}{b^1{}_1}b^2{}_0\,, \qquad b^2{}_2{}' + \omega^0{}_2 b^1{}_1 = 0\,, \\
&\frac{2}{\ell} = \frac{\omega^2{}_2}{b^2{}_2} + \frac{\omega^1{}_1}{b^1{}_1}\,, \qquad \frac{2}{\ell} + \omega^0{}_2\frac{b^2{}_0}{b^0{}_0 b^2{}_2} = \frac{\omega^2{}_2}{b^2{}_2} + \frac{\omega^0{}_0}{b^0{}_0}\,, \\
&\frac{2}{\ell} - \frac{b^2{}_0{}'}{b^0{}_0 b^1{}_1} = \frac{\omega^1{}_1}{b^1{}_1} + \frac{\omega^0{}_0}{b^0{}_0}\,.
\end{aligned}$$

Solving these equations we find

$$\begin{aligned}
&\omega^0 = N dx^-\,, \qquad \omega^1 = \left(\frac{1}{\ell} + \frac{J}{r^2}\right)\frac{1}{N}dr\,, \\
&\omega^2 = -r\left(\frac{1}{\ell} - \frac{J}{r^2}\right)dx^- - \frac{J}{2r^3}dt\,.
\end{aligned} \tag{3.6}$$

This completes the derivation of the exact black hole solution in the telaparallel geometry. The AdS vacuum solution (3.2), (3.3) is obtained for $2m = -1$, $J = 0$. For other values of m and J, the black hole differs from the AdS vacuum, but has similar asymptotic behaviour.

IV. ASYMPTOTIC CONDITIONS

Dynamical structure of a field theory is determined not only by the field equations, but also by the asymptotic conditions. An important feature of this structure is contained in its symmetry properties. When $\Lambda_{\text{eff}} < 0$, the solution of (2.5) possessing the maximum number of symmetries is the AdS solution [20]. It plays the role analogous to the role of Minkowski space in the $\Lambda_{\text{eff}} = 0$ case. Therefore, it seems natural to choose the asymptotic behaviour in such a way that all the dynamical variables approach the AdS configuration at large distances. On the other hand, such an approach would exclude the important (locally equivalent but globally distinct) black hole geometries. Then again, these geometries are not AdS invariant — the minimal feature we would like to have.

Having this in mind, the concept of the *AdS asymptotic behaviour* can be defined by imposing the following requirements [1,23]:

a) the asymptotic conditions should be invariant under the action of the AdS group;
b) they should include the important black hole geometries;
c) the asymptotic symmetries should have well defined canonical generators.

The first two requirements are studied in this section, while c) is left for the next two sections.

A. Asymptotic AdS configurations

We begin our considerations with the point b) above. The asymptotic behaviour of the black hole solution is easily derived from equations (3.5) and (3.6). For the triad field, it is given by

$$b^i{}_\mu \sim \begin{pmatrix} \dfrac{r}{\ell} - \dfrac{m\ell}{r} & 0 & 0 \\ 0 & \dfrac{\ell}{r} + \dfrac{m\ell^3}{r^3} & 0 \\ \dfrac{J}{r} & 0 & r \end{pmatrix} . \tag{4.1a}$$

For the simplicity of notation, the type of higher order terms on the right hand side is not written explicitly. Similarly, the asymptotic behaviour of the connection has the form

$$\omega^i{}_\mu \sim \begin{pmatrix} \dfrac{r}{\ell^2} - \dfrac{m}{r} & 0 & -\dfrac{r}{\ell} + \dfrac{m\ell}{r} \\ 0 & \dfrac{1}{r} + \dfrac{J\ell + m\ell^2}{r^3} & 0 \\ -\dfrac{r}{\ell^2} + \dfrac{J}{\ell r} & 0 & \dfrac{r}{\ell} - \dfrac{J}{r} \end{pmatrix} . \tag{4.1b}$$

The requirement b) means that the asymptotic behaviour must be such as to allow for the black hole configuration (4.1).

In the next step, we turn to the requirement a). It can be realized by starting with the black hole configuration (4.1) and acting on it with all possible AdS transformations. Instead of that, we shall use the known result of such a procedure for the black hole metric, and then transform the obtained information to the triad and connection.

The family of metrics obtained by acting on the black hole metric (3.4) with all AdS transformations, has been found by Brown and Henneaux [1]:

$$g_{\mu\nu} = \begin{pmatrix} \dfrac{r^2}{\ell^2} + \mathcal{O}_0 & \mathcal{O}_3 & \mathcal{O}_0 \\ \mathcal{O}_3 & -\dfrac{\ell^2}{r^2} + \mathcal{O}_4 & \mathcal{O}_3 \\ \mathcal{O}_0 & \mathcal{O}_3 & -r^2 + \mathcal{O}_0 \end{pmatrix} \equiv \begin{pmatrix} \dfrac{r^2}{\ell^2} & 0 & 0 \\ 0 & -\dfrac{\ell^2}{r^2} & 0 \\ 0 & 0 & -r^2 \end{pmatrix} + G_{\mu\nu} ,$$

where $\mathcal{O}_n$ denotes a quantity that tends to zero as $1/r^n$ or faster, when $r \to \infty$. The set of AdS transformations is defined by six Killing vectors (Appendix A), hence, strictly speaking, the set of all metrics obtained from (3.4) by the action of these transformations is parametrized by six real parameters, say σ_i. The meaning of the above expression for $g_{\mu\nu}$ is slightly different: any c/r^n term it supposed to be of the form $c(t,\varphi)/r^n$, i.e. constants $c = c(\sigma_i)$ of the six parameter family are promoted to functions $c(t,\varphi)$. This is the simplest way to characterize the asymptotic behaviour of the family $g_{\mu\nu}$.

In accordance with the above result, we adopt the following asymptotic form for the triad field:

$$b^i{}_\mu = \begin{pmatrix} \dfrac{r}{\ell} + \mathcal{O}_1 & O_4 & O_1 \\ \mathcal{O}_2 & \dfrac{\ell}{r} + \mathcal{O}_3 & O_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & r + \mathcal{O}_1 \end{pmatrix} \equiv \begin{pmatrix} \dfrac{r}{\ell} & 0 & 0 \\ 0 & \dfrac{\ell}{r} & 0 \\ 0 & 0 & r \end{pmatrix} + B^i{}_\mu . \tag{4.2a}$$

It generates the Brown–Henneaux asymptotic behaviour of the metric, but is not uniquely determined by it. Indeed, we can apply an arbitrary local Lorentz transformation to (4.2a), thereby changing its asymptotics, but it will not affect the metric in any way. Our choice of the triad asymptotics was guided by two principles: i) to obtain as general asymptotic behaviour as possible, and ii) to evade additional constraint relations among (otherwise arbitrary) higher order terms $B^i{}_\mu$.

Next, we use the torsion equation of motion (2.7a) to obtain the asymptotic form of the connection:

$$\omega^i{}_\mu = \begin{pmatrix} \frac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_2 & -\frac{r}{\ell} + \mathcal{O}_1 \\ \mathcal{O}_2 & \frac{1}{r} + \mathcal{O}_3 & \mathcal{O}_2 \\ -\frac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_2 & \frac{r}{\ell} + \mathcal{O}_1 \end{pmatrix} \equiv \begin{pmatrix} \frac{r}{\ell^2} & 0 & -\frac{r}{\ell} \\ 0 & \frac{1}{r} & 0 \\ -\frac{r}{\ell^2} & 0 & \frac{r}{\ell} \end{pmatrix} + \Omega^i{}_\mu \,. \tag{4.2b}$$

Again, the higher order terms $\Omega^i{}_\mu$ are considered arbitrary and independent of those in (4.2a). One can check that the asymptotic conditions (4.2) are indeed invariant under the action of the AdS group.

B. Asymptotic symmetries

We are now going to examine the symmetries of the asymptotic conditions (4.2). The parameters of gauge transformations that leave the conditions (4.2) invariant are determined by the relations

$$\varepsilon^{ijk}\theta_j b_{k\mu} - \xi^\rho{}_{,\mu} b^i{}_\rho - \xi^\rho \partial_\rho b^i{}_\mu = \delta_0 B^i{}_\mu \,, \tag{4.3a}$$
$$-\theta^i{}_{,\mu} + \varepsilon^{ijk}\theta_j \omega_{k\mu} - \xi^\rho{}_{,\mu}\omega^i{}_\rho - \xi^\rho \partial_\rho \omega^i{}_\mu = \delta_0 \Omega^i{}_\mu \,. \tag{4.3b}$$

Acting on a specific field satisfying the adopted asymptotic conditions, these transformations change the form of the non-leading terms $B^i{}_\mu$, $\Omega^i{}_\mu$. One should stress that the symmetry transformations defined in this way differ from the usual asymptotic symmetries, which act according to the rule $\delta_0 b^i{}_\mu = 0$, $\delta_0\omega^i{}_\mu = 0$.

We shall find the gauge parameters in three steps.

1. The symmetric part of the first equation multiplied by $b_{i\nu}$ (six relations) has the form

$$-\xi^\rho{}_{,\mu} g_{\nu\rho} - \xi^\rho{}_{,\nu} g_{\mu\rho} - \xi^\rho \partial_\rho g_{\mu\nu} = \delta_0 G_{\mu\nu} \,, \tag{4.4}$$

which defines the transformation rule of the metric. The matrix $\delta_0 G_{\mu\nu}$ has the same form as $G_{\mu\nu}$. If we define the expansion of ξ^μ in powers of r^{-1},

$$\xi^\mu = \sum_{n=-1}^{\infty} \xi^\mu_n r^{-n} \,,$$

the condition (4.4) yields

$$\begin{aligned} &\xi^0_{-1} = \xi^0_1 = \xi^0_3 = 0 \,, \qquad \xi^1_0 = 0 \,, \qquad \xi^2_{-1} = \xi^2_1 = \xi^2_3 = 0 \,, \\ &\xi^0_2 = \frac{\ell^4}{2}\,\xi^0_{0,00} \,, \qquad \xi^1_{-1} = -\xi^0_{0,0} \,, \qquad \xi^2_2 = -\frac{\ell^2}{2}\,\xi^2_{0,22} \,, \\ &\xi^2_{0,2} = \xi^0_{0,0} \,, \qquad \xi^2_{0,0} = \frac{1}{\ell^2}\,\xi^0_{0,2} \,. \end{aligned}$$

Then, after introducing the notation

$$\xi_0^0 = \ell T(t,\varphi)\,, \qquad \xi_0^2 = S(t,\varphi)\,,$$

the solution of the above equations takes the form

$$\xi^0 = \ell\left[T + \frac{1}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)\frac{\ell^4}{r^2}\right] + \mathcal{O}_4\,, \tag{4.5a}$$

$$\xi^2 = S - \frac{1}{2}\left(\frac{\partial^2 S}{\partial \varphi^2}\right)\frac{\ell^2}{r^2} + \mathcal{O}_4\,, \tag{4.5b}$$

$$\xi^1 = -\ell\left(\frac{\partial T}{\partial t}\right) r + \mathcal{O}_1\,, \tag{4.5c}$$

where the functions $T(t,\varphi)$ and $S(t,\varphi)$ satisfy the conditions

$$\frac{\partial T}{\partial \varphi} = \ell\frac{\partial S}{\partial t}\,, \qquad \frac{\partial S}{\partial \varphi} = \ell\frac{\partial T}{\partial t}\,. \tag{4.6}$$

The above equations define the conformal group of transformations at large distances [1]. Whether this group will survive as the asymptotic symmetry of our teleparallel theory depends on the remaining conditions in (4.3).

2. After having used six components of (4.3a) to find the form of ξ^μ, we shall now determine θ^i from the remaining three components. They yield the relations

$$\begin{aligned}
&\theta^1 b^2{}_1 - \theta^2 b^1{}_1 - \xi^0{}_{,1} b^0{}_0 = \mathcal{O}_4\,, \qquad \theta^1 b^2{}_2 - \xi^0{}_{,2} b^0{}_0 = \mathcal{O}_1\,,\\
&\theta^0 b^2{}_2 - \xi^\rho{}_{,2} b^1{}_\rho = \mathcal{O}_2\,,
\end{aligned}$$

with the solution

$$\begin{aligned}
\theta^0 &= -\frac{\ell^2}{r} T_{,02} + \mathcal{O}_3\,,\\
\theta^2 &= \frac{\ell^3}{r} T_{,00} + \mathcal{O}_3\,,\\
\theta^1 &= T_{,2} + \mathcal{O}_2\,.
\end{aligned} \tag{4.5d}$$

3. We have seen that the invariance conditions (4.3a) completely define the parameters ξ^μ and θ^i, as shown in (4.5). Now, the last thing to check is whether the transformation law for the connection leads to any new limitation on the parameters. Explicit calculation shows that the connection configuration (4.2b) is also invariant under the transformations defined by the above parameters.

The allowed form of the parameters T and S is obtained by solving equations (4.6). Rewriting these conditions in the form

$$\partial_\pm(T \mp S) = 0\,,$$

with $2\partial_\pm = \ell\partial_0 \pm \partial_2$, we find that the general solution is given by

$$T + S = f(x^+)\,, \qquad T - S = g(x^-)\,, \tag{4.7}$$

where f and g are two arbitrary, periodic functions.

We have found that our gauge parameters (ξ^μ, θ^i) must be of the form (4.5), in order to preserve the adopted boundary conditions (4.2). At large distances, the parameters (ξ^μ, θ^i) are determined by the functions (T, S) which define the conformal symmetry at the boundary of our teleparallel spacetime. (This will be shown in Sec. VI, while the related analysis in Riemannian case is given in Ref. [1].) However, the complete gauge group defined in this way contains also the *residual* (or pure) gauge transformations, characterized by the higher $\mathcal{O}_n$ terms in (4.5), the only terms that remain after imposing $T = S = 0$. As we shall see, the residual gauge transformations do not contribute to the values of the conserved charges, and consequently, their generators vanish weakly. In order to "eliminate" the residual gauge transformations from our discussion, we introduce the concept of *asymptotic symmetry* in the following way:

> the asymptotic symmetry group is defined as the factor group of the gauge group determined by (4.5), with respect to the residual gauge group.

In other words, two asymptotic symmetry transformations are identified if their parameters (ξ^μ, θ^i) have identical (T, S) pairs, while any difference stemming from the higher $\mathcal{O}_n$ terms in (4.5) is ignored.

In conclusion, the adopted asymptotic behaviour (4.2) defines the most general configuration space of the theory that respects our requirements a) and b) formulated at the beginning of this section. The related symmetry structure is determined by the parameters (4.5). In order to verify the status of the last requirement c), it is necessary to explore the canonical structure of the theory.

V. HAMILTONIAN STRUCTURE

Gauge symmetries of a dynamical system are best described by the canonical generators. After clarifying the canonical and gauge structure of our theory of gravity, we shall be able to better understand the meaning of the adopted asymptotic conditions.

The action of the theory (2.9) can equivalently be written as

$$I = a\int d^3x \varepsilon^{\rho\mu\nu}\left[b^i{}_\rho\left(R_{i\mu\nu} - \frac{1}{\ell}T_{i\mu\nu}\right) + \frac{4}{3\ell^2}\,\varepsilon_{ijk}b^i{}_\rho b^j{}_\mu b^k{}_\nu\right]. \tag{5.1}$$

A. Hamiltonian and constraints

Starting from the definition of the the canonical momenta $(\pi_i{}^\mu, \Pi_i{}^\mu)$, corresponding to the basic Lagrangian variables $(b^i{}_\mu, \omega^i{}_\mu)$, we find the following primary constraints:

$$\begin{aligned}
&\phi_i{}^0 \equiv \pi_i{}^0 \approx 0\,, \qquad \Phi_i{}^0 \equiv \Pi_i{}^0 \approx 0\,, \\
&\phi_i{}^\alpha \equiv \pi_i{}^\alpha + \frac{2a}{\ell}\varepsilon^{0\alpha\beta}b_{i\beta} \approx 0\,, \qquad \Phi_i{}^\alpha \equiv \Pi_i{}^\alpha - 2a\varepsilon^{0\alpha\beta}b_{i\beta} \approx 0\,.
\end{aligned} \tag{5.2}$$

Since the Lagrangian is linear in velocities, the canonical Hamiltonian is determined by the formula $\mathcal{H}_c = -\mathcal{L}(\dot{b}^i{}_\mu = \dot{\omega}^i{}_\mu = 0)$. It is linear in unphysical variables $(b^i{}_0, \omega^i{}_0)$, as we expect:

$$\mathcal{H}_c = b^i{}_0\mathcal{H}_i + \omega^i{}_0\mathcal{K}_i + \partial_\alpha D^\alpha ,$$

where

$$\begin{aligned}
\mathcal{H}_i &= -a\varepsilon^{0\alpha\beta}\left(R_{i\alpha\beta} - \frac{2}{\ell}T_{i\alpha\beta} + \frac{4}{\ell^2}\,\varepsilon_{ijk}b^j{}_\alpha b^k{}_\beta\right) , \\
\mathcal{K}_i &= -a\varepsilon^{0\alpha\beta}\left(T_{i\alpha\beta} - \frac{2}{\ell}\varepsilon_{ijk}b^j{}_\alpha b^k{}_\beta\right) , \\
D^\alpha &= 2a\varepsilon^{0\alpha\beta}b^i{}_\beta\left(\omega_{i0} - \frac{1}{\ell}b_{i0}\right) .
\end{aligned}$$

The total Hamiltonian has the form

$$\mathcal{H}_T = b^i{}_0\mathcal{H}_i + \omega^i{}_0\mathcal{K}_i + u^i{}_\mu\phi_i{}^\mu + v^i{}_\mu\Phi_i{}^\mu + \partial_\alpha D^\alpha . \tag{5.3}$$

The consistency conditions of the sure primary constraints $\pi_i{}^0$ and $\Pi_i{}^0$ lead to the secondary constraints:

$$\mathcal{H}_i \approx 0 , \qquad \mathcal{K}_i \approx 0 , \tag{5.4}$$

which can be equivalently written as

$$R_{i\alpha\beta} \approx 0 , \qquad T_{i\alpha\beta} \approx \frac{2}{\ell}\varepsilon_{imn}b^m{}_\alpha b^n{}_\beta .$$

The consistency of the remaining primary constraints $\phi_i{}^\alpha$ and $\Phi_i{}^\alpha$ leads to the determination of the multipliers $u^i{}_\beta$ and $v^i{}_\beta$:

$$\begin{aligned}
&u^i{}_\beta + \varepsilon^{imn}\omega_{m0}b_{n\beta} - \nabla_\beta b^i{}_0 = \frac{2}{\ell}\varepsilon^{imn}b_{m0}b_{n\beta} , \\
&v^i{}_\beta - \nabla_\beta\omega^i{}_0 = 0 .
\end{aligned} \tag{5.5}$$

Since the equation of motion for $b^i{}_\beta$ has the form $u^i{}_\beta = \dot{b}^i{}_\beta$, the first relation can be written as the field equation $T^i{}_{0\beta} = (2/\ell)\varepsilon^{imn}b_{m0}b_{n\beta}$, with $\dot{b}^i{}_\beta \to u^i{}_\beta$. Similarly, the second relation is on shell equivalent to the field equation $R^i{}_{0\beta} = 0$, with $\dot{\omega}^i{}_\beta \to v^i{}_\beta$. The result is obtained using the following Poisson brackets (PBs) involving $\phi_i{}^\alpha$ and $\Phi_i{}^\alpha$:

$$\begin{aligned}
&\{\phi_i{}^\alpha, \phi_j{}^\beta\} = \frac{4a}{\ell}\varepsilon^{0\alpha\beta}\eta_{ij}\delta , \\
&\{\phi_i{}^\alpha, \Phi_j{}^\beta\} = -2a\varepsilon^{0\alpha\beta}\eta_{ij}\delta , \qquad \{\Phi_i{}^\alpha, \Phi_j{}^\beta\} = 0 ,
\end{aligned}$$

and

$$\begin{aligned}
\{\phi_i{}^\alpha, \mathcal{H}_j\} &= -\frac{2}{\ell}\{\phi_i{}^\alpha, \mathcal{K}_j\} = -\frac{2}{\ell}\{\Phi_i{}^\alpha, \mathcal{H}_j\} , \\
\{\Phi_i{}^\alpha, \mathcal{H}_j\} &= 2a\varepsilon^{0\alpha\beta}\left[\eta_{ij}\partial_\beta\delta - \varepsilon_{ijn}\left(\omega^n{}_\beta - \frac{2}{\ell}b^n{}_\beta\right)\delta\right] , \\
\{\Phi_i{}^\alpha, \mathcal{K}_j\} &= -2a\varepsilon^{0\alpha\beta}\varepsilon_{ijn}b^n{}_\beta\delta .
\end{aligned}$$

Here, we use the notation $\{A,B\}=\{A(x),B(y)\}$, $\delta=\delta(x-y)$, $\partial_\beta=\partial/\partial x^\beta$.

Replacing the expressions (5.5) into the total Hamiltonian (5.3), we obtain

$$
\begin{aligned}
&\mathcal{H}_T=\hat{\mathcal{H}}_T+\partial_\alpha\bar{D}^\alpha\,,\\
&\hat{\mathcal{H}}_T=b^i{}_0\bar{\mathcal{H}}_i+\omega^i{}_0\bar{\mathcal{K}}_i+u^i{}_0\pi_i{}^0+v^i{}_0\Pi_i{}^0\,,
\end{aligned}
\tag{5.6a}
$$

where

$$
\begin{aligned}
&\bar{\mathcal{H}}_i=\mathcal{H}_i-\nabla_\beta\phi_i{}^\beta+\frac{2}{\ell}\varepsilon_{imn}b^m{}_\beta\phi^{n\beta}\,,\\
&\bar{\mathcal{K}}_i=\mathcal{K}_i-\nabla_\beta\Phi_i{}^\beta-\varepsilon_{imn}b^m{}_\beta\phi^{n\beta}\,,\\
&\bar{D}^\alpha=D^\alpha+b^k{}_0\phi_k{}^\alpha+\omega^k{}_0\Phi_k{}^\alpha\,.
\end{aligned}
\tag{5.6b}
$$

Further investigation of the consistency requirements is facilitated by observing that the secondary constraints $\bar{\mathcal{H}}_i,\bar{\mathcal{K}}_i$ obey the following PB algebra:

$$
\begin{aligned}
&\{\bar{\mathcal{H}}_i,\bar{\mathcal{H}}_j\}=\frac{2}{\ell}\varepsilon_{ijk}\bar{\mathcal{H}}^k\delta\,,\qquad \{\bar{\mathcal{H}}_i,\bar{\mathcal{K}}_j\}=-\varepsilon_{ijk}\bar{\mathcal{H}}^k\delta\,,\\
&\{\bar{\mathcal{K}}_i,\bar{\mathcal{K}}_j\}=-\varepsilon_{ijk}\bar{\mathcal{K}}^k\delta\,.
\end{aligned}
\tag{5.7}
$$

Indeed, we can now conclude that consistency conditions of the secondary constraints are identically satisfied, which completes the consistency procedure.

The complete dynamical classification of the constraints is given in Table 1.

Table 1. Classification of constraints

	first class	second class
primary	$\pi_i{}^0,\Pi_i{}^0$	$\phi_i{}^\alpha,\Phi_i{}^\alpha$
secondary	$\bar{\mathcal{H}}_i,\bar{\mathcal{K}}_i$	

We display here, for completeness, the PBs between $(\phi_i{}^\alpha,\Phi_i{}^\alpha)$ and $(\bar{\mathcal{H}}_j,\bar{\mathcal{K}}_j)$:

$$
\begin{aligned}
&\{\phi_i{}^\alpha,\bar{\mathcal{H}}_j\}=\frac{2}{\ell}\,\varepsilon_{ijk}\phi^{k\alpha}\delta\,,\qquad \{\phi_i{}^\alpha,\bar{\mathcal{K}}_j\}=-\varepsilon_{ijk}\phi^{k\alpha}\delta\,,\\
&\{\Phi_i{}^\alpha,\bar{\mathcal{H}}_j\}=-\varepsilon_{ijk}\phi^{k\alpha}\delta\,,\qquad \{\Phi_i{}^\alpha,\bar{\mathcal{K}}_j\}=-\varepsilon_{ijk}\Phi^{k\alpha}\delta\,.
\end{aligned}
$$

B. Gauge generators

The presence of arbitrary multipliers in the total Hamiltonian indicates the existence of gauge symmetries in the theory. The canonical gauge generators can be constructed using the general method of Castellani [24].

Starting from the primary first class constraints $\pi_i{}^0$ and $\Pi_i{}^0$, we find the form of the respective gauge generators:

$$
\begin{aligned}
&G[\epsilon]=\dot{\epsilon}^i\pi_i{}^0+\epsilon^i\left[\bar{\mathcal{H}}_i-\varepsilon_{ijk}\left(\omega^j{}_0-\frac{2}{\ell}\,b^j{}_0\right)\pi^{k0}\right]\,,\\
&G[\tau]=\dot{\tau}^i\Pi_i{}^0+\tau^i\left[\bar{\mathcal{K}}_i-\varepsilon_{ijk}\left(b^j{}_0\,\pi^{k0}+\omega^j{}_0\,\Pi^{k0}\right)\right]\,.
\end{aligned}
\tag{5.8}
$$

The complete gauge generator is given as the sum $G = G[\epsilon] + G[\tau]$, and it produces the following gauge transformations on the fields ($\delta_0\phi \equiv \{\phi\,, G\}$):

$$
\begin{aligned}
\delta_0\, b^i{}_\mu &= \nabla_\mu \epsilon^i - \frac{2}{\ell}\,\varepsilon^i{}_{jk} b^j{}_\mu \epsilon^k + \varepsilon^i{}_{jk} b^j{}_\mu \tau^k\,,\\
\delta_0\, \omega^i{}_\mu &= \nabla_\mu \tau^i\,.
\end{aligned}
$$

Introducing now the new parameters ξ^μ and θ^i,

$$
\epsilon^i = -\xi^\mu b^i{}_\mu\,, \qquad \tau^i = -(\theta^i + \xi^\mu \omega^i{}_\mu)\,, \tag{5.9}
$$

the transformation law takes the form

$$
\begin{aligned}
\delta_0 b^i{}_\mu &= \varepsilon^{ijk}\theta_j b_{k\mu} - \xi^\rho{}_{,\mu} b^i{}_\rho - \xi^\rho b^i{}_{\mu,\,\rho} - \xi^\rho \left(T^i{}_{\mu\rho} - \frac{2}{\ell}\,\varepsilon^{ijk} b_{j\mu} b_{k\rho}\right)\,,\\
\delta_0 \omega^i{}_\mu &= -\nabla_\mu \theta^i - \xi^\rho{}_{,\mu}\omega^i{}_\rho - \xi^\rho \omega^i{}_{\mu,\,\rho} - \xi^\rho R_{i\mu\rho}\,,
\end{aligned}
$$

which is in complete agreement with the transformations (2.1a) *on-shell*. The gauge generator G, expressed in terms of the new parameters ξ^μ and θ^i, is given by

$$
\begin{aligned}
G &= -G_1 - G_2\,,\\
G_1 &\equiv \dot{\xi}^\rho \left(b^i{}_\rho \pi_i{}^0 + \omega^i{}_\rho \Pi_i{}^0 \right) + \xi^\rho \left[b^i{}_\rho \bar{\mathcal{H}}_i + \omega^i{}_\rho \bar{\mathcal{K}}_i + (\partial_\rho b^i{}_0)\pi_i{}^0 + (\partial_\rho \omega^i{}_0)\Pi_i{}^0 \right]\,,\\
G_2 &\equiv \dot{\theta}^i \Pi_i{}^0 + \theta^i \left[\bar{\mathcal{K}}_i - \varepsilon_{ijk}\left(b^j{}_0 \pi^{k0} + \omega^j{}_0\, \Pi^{k0}\right)\right]\,.
\end{aligned} \tag{5.10}
$$

Here, the time derivatives $\dot{b}^i{}_\mu$ and $\dot{\omega}^i{}_\mu$ are shorts for $u^i{}_\mu$ and $v^i{}_\mu$, respectively. The result is obtained by discarding terms that produce trivial transformations on-shell. Note that the time translation generator $G_1[\xi^0]$ is defined in terms of $\hat{\mathcal{H}}_T$:

$$
G_1[\xi^0] = \dot{\xi}^0 \left(b^i{}_0 \pi_i{}^0 + \omega^i{}_0 \Pi_i{}^0 \right) + \xi^0 \hat{\mathcal{H}}_T\,.
$$

(In the above expressions for the gauge generators, we did not write the integration symbol $\int d^2x$ in order to simplify the notation. Later, where necessary, the integration symbol will be restored.)

To complete the analysis of the asymptotic structure of phase space, we shall now define the behaviour of momentum variables at large distances. Our procedure is based on the following general principle: the expressions that vanish on-shell should have an *arbitrarily fast asymptotic decrease*, as no solution of the field equations is thereby lost. Applied to the primary constraints of the theory, this principle gives us the asymptotic behaviour of the momenta $\pi_i{}^\mu$ and $\Pi_i{}^\mu$:

$$
\begin{aligned}
\pi_i{}^0 &= \hat{\mathcal{O}}\,, \qquad & \pi_i{}^\alpha &= -\frac{2a}{\ell}\varepsilon^{0\alpha\beta} b_{i\beta} + \hat{\mathcal{O}}\,,\\
\Pi_i{}^0 &= \hat{\mathcal{O}}\,, \qquad & \Pi_i{}^\alpha &= 2a\,\varepsilon^{0\alpha\beta} b_{i\beta} + \hat{\mathcal{O}}\,,
\end{aligned} \tag{5.11}
$$

where $\hat{\mathcal{O}}$ denotes a term with arbitrarily fast decrease. The asymptotic form of the secondary constraints, as well as some of the equations of motion, are given in Appendix B. They further refine the asymptotic behaviour of phase space variables, and serve as a tool to prove the finiteness of our improved gauge generators.

We are now ready to discuss the impact of the adopted boundary conditions on the form of the canonical generator.

VI. ASYMPTOTIC SYMMETRY

After having introduced the notion of the asymptotic symmetry group in section IV, we now wish to continue with the related canonical analysis. For the asymptotic generator, we use the simple notation $G[T,S]$, indicating the irrelevance of the residual gauge parameters in (4.5). (Although the concept of an asymptotic generator is different from the gauge generator in general, we use the same symbol G for both, in order to keep the notation simple.)

In what follows, we shall construct the improved form of the general gauge generator G, assuming the asymptotic conditions (4.2) and (4.5), and prove the conservation of the corresponding charges; then, we shall investigate the form and properties of the canonical algebra of asymptotic generators, and find the value of the related central charge.

A. Boundary terms

In the Hamiltonian theory, the generators of symmetry transformations act on dynamical variables via the PB operation, which is defined in terms of functional derivatives. A functional $F[\varphi,\pi]=\int d^3x f(\varphi,\partial_\mu\varphi,\pi,\partial_\nu\pi)$ has well defined functional derivatives if its variation can be written in the form

$$\delta F=\int d^3x[A(x)\delta\varphi(x)+B(x)\delta\pi(x)]\,,$$

where terms $\delta\varphi_{,\mu}$ and $\delta\pi_{,\nu}$ are absent. In addition, the well defined phase space functionals have to be finite.

Our general gauge generator G does not meet these requirements, although when acting on local expressions, it produces the correct transformation laws. We shall now improve its form by adding an appropriate surface term, so that it can also act on global phase space functionals [18].

We begin by considering the variation of G_2:

$$\begin{aligned}\delta G_2&=\theta^i\delta\bar{\mathcal{K}}_i+R=\theta^i\delta\mathcal{K}_i+\partial\hat{\mathcal{O}}+R\\&=-2a\varepsilon^{\alpha\beta0}\theta^i\partial_\alpha\delta b_{i\beta}+\partial\hat{\mathcal{O}}+R\\&=-2a\varepsilon^{\alpha\beta0}\partial_\alpha\left(\theta^i\delta b_{i\beta}\right)+\partial\hat{\mathcal{O}}+R=\partial\mathcal{O}_2+R\,.\end{aligned}$$

The last equality is a consequence of the relation $\theta^i\delta b_{i\beta}=\mathcal{O}_2$, derived from the asymptotic conditions (4.5d) and (4.2). The total divergence term $\partial\mathcal{O}_2$ gives a vanishing contribution after integration, as follows from the Stokes theorem:

$$\int_{\mathcal{M}_2}d^2x\,\partial_\alpha v^\alpha=\int_{\partial\mathcal{M}_2}v^\alpha df_\alpha=\int_0^{2\pi}v^1d\varphi\qquad(df_\alpha\equiv\varepsilon_{\alpha\beta}dx^\beta)\,.$$

In the last equality, the boundary of $\mathcal{M}_2$ is taken to be the circle at infinity, parametrized by the angular coordinate φ. Thus, G_2 is a regular generator for which there is no need to introduce any boundary term:

$$\text{Boundary term for } G_2=0\,.\qquad(6.1)$$

The variation of G_1 yields

$$
\begin{aligned}
\delta G_1 &= \xi^\rho \left(b^i{}_\rho \delta\bar{\mathcal{H}}_i + \omega^i{}_\rho \delta\bar{\mathcal{K}}_i\right) + \partial\hat{\mathcal{O}} + R \\
&= -2a\varepsilon^{\alpha\beta 0}\partial_\alpha \left[\xi^\rho b^i{}_\rho \delta\left(\omega_{i\beta} - \frac{2}{\ell} b_{i\beta}\right) + \xi^\rho \omega^i{}_\rho \delta b_{i\beta}\right] + \partial\hat{\mathcal{O}} + R\,.
\end{aligned}
$$

We shall now focus on the terms containing ξ^0, ξ^1 and ξ^2.

1. Using the preceding result and the asymptotic conditions (4.5), (4.2), the variation of the generator $G_1[\xi^0]$ is shown to have the form

$$
\delta G_1[\xi^0] = -2a\varepsilon^{\alpha\beta 0}\partial_\alpha \left[\xi^0 b^0{}_0 \delta\left(\omega^0{}_\beta - \frac{1}{\ell} b^0{}_\beta + \frac{1}{\ell} b^2{}_\beta\right)\right] + \partial\mathcal{O}_2 + R\,,
$$

wherefrom

$$
\begin{aligned}
&\delta G_1[\xi^0] = -\delta\partial_\alpha\left(\xi^0 \mathcal{E}^\alpha\right) + \partial\mathcal{O}_2 + R\,, \\
&\mathcal{E}^\alpha = 2a\,\varepsilon^{\alpha\beta 0}\left(\omega^0{}_\beta + \frac{1}{\ell}\, b^2{}_\beta - \frac{1}{\ell}\, b^0{}_\beta\right) b^0{}_0\,.
\end{aligned}
$$

The improved time translation generator reads:

$$
\begin{aligned}
&\tilde{G}_1[\xi^0] = G_1[\xi^0] + E[\xi^0]\,, \\
&E[\xi^0] = \oint \xi^0 \mathcal{E}^\alpha df_\alpha = \int_0^{2\pi} d\varphi\, \xi^0 \mathcal{E}^1\,.
\end{aligned}
\tag{6.2a}
$$

From $\xi^0 = \mathcal{O}_0$, $\mathcal{E}^1 = \mathcal{O}_0$, it follows that E must be finite.

Note that, for $\xi^0 = 1$, the generator G_1 reduces to $\hat{\mathcal{H}}_T$, so that the corresponding boundary term defines the improved Hamiltonian, and has the meaning of energy:

$$
\begin{aligned}
&\tilde{H}_T = \hat{H}_T + E_0\,, \\
&E_0 = \oint \mathcal{E}^\alpha df_\alpha = \int_0^{2\pi} \mathcal{E}^1 d\varphi\,.
\end{aligned}
\tag{6.2b}
$$

2. In a similar way, we find that the variation of $G_1[\xi^2]$ has the form

$$
\delta G_1[\xi^2] = 2a\,\varepsilon^{\alpha\beta 0}\partial_\alpha \left[\xi^2 b^2{}_2 \delta\left(\omega^2{}_\beta + \frac{1}{\ell} b^0{}_\beta - \frac{1}{\ell} b^2{}_\beta\right)\right] + \partial\mathcal{O}_2 + R\,,
$$

whereupon we conclude that

$$
\begin{aligned}
&\delta G_1[\xi^2] = -\delta\partial_\alpha\left(\xi^2 \mathcal{M}^\alpha\right) + \partial\mathcal{O}_2 + R\,, \\
&\mathcal{M}^\alpha = -2a\,\varepsilon^{\alpha\beta 0}\left(\omega^2{}_\beta + \frac{1}{\ell}\, b^0{}_\beta - \frac{1}{\ell}\, b^2{}_\beta\right) b^2{}_2\,.
\end{aligned}
$$

The improved spatial rotation generator reads:

$$
\begin{aligned}
&\tilde{G}_1[\xi^2] = G_1[\xi^2] + M[\xi^2]\,, \\
&M[\xi^2] = \oint \xi^2 \mathcal{M}^\alpha df_\alpha = \int_0^{2\pi} d\varphi\, \xi^2 \mathcal{M}^1\,.
\end{aligned}
\tag{6.3a}
$$

The boundary term

$$M_0 = \oint \mathcal{M}^\alpha df_\alpha = \int_0^{2\pi} \mathcal{M}^1 d\varphi \tag{6.3b}$$

represents the angular momentum of the system. The adopted asymptotics ensures the finiteness of the improved generator.

3. Analogous considerations in the case of $G_1[\xi^1]$ lead us to the conclusion that this generator is regular,

$$\text{Boundary term for } G_1[\xi^1] = 0\,. \tag{6.4}$$

Therefore, the improved gauge generator $\tilde{G}$ is given by the expression

$$\begin{aligned} &\tilde{G} = G + \Gamma\,, \\ &\Gamma = -\oint df_\alpha\Big(\xi^0\mathcal{E}^\alpha + \xi^2\mathcal{M}^\alpha\Big) = -\int_0^{2\pi} d\varphi\Big(\ell T\mathcal{E}^1 + S\mathcal{M}^1\Big)\,. \end{aligned} \tag{6.5}$$

The adopted asymptotic behaviour of fields and parameters guarantees finiteness of the surface term Γ. Being a linear combination of constraints, the generator G is finite itself, and therefore, the improved generator $\tilde{G}$ is well defined, differentiable functional on its whole domain.

As we can see, the surface term Γ depends only on the parameters (T, S), and not on the higher order terms in (4.5). Thus, it is only the *asymptotic* generators that have non-trivial boundary terms, and, consequently, do not vanish weakly. We expect the corresponding conserved charges to be physically non-trivial. On the other hand, the *residual* gauge generators are characterized by vanishing Γ, and can only have zero charges [1,22].

B. Canonical algebra

We now wish to find the form of the canonical algebra of the improved asymptotic generators, which contains important information on the symmetry structure of the asymptotic dynamics. We will use this algebra to prove the conservation of the boundary terms. Introducing the notation

$$G' \equiv G[T', S']\,, \qquad G'' \equiv G[T'', S'']\,, \quad \dots\,,$$

(similarly $\delta_0' \equiv \delta_0[T', S']$, etc.) we find that

$$\Big\{\tilde{G}'', \tilde{G}'\Big\} = \delta_0'\tilde{G}'' \approx \delta_0'\Gamma''\,, \tag{6.6}$$

because every symmetry generator commutes with all the constraints of the theory. The right-hand side of the above equation represents the transformation of the surface term Γ'' under the action of the generator $\tilde{G}'$. Using the transformation rules (2.1a) with parameters (4.5), and refined asymptotic conditions (B3a) of Appendix B, we find:

$$\begin{aligned} &\delta_0(\ell\mathcal{E}^1) = -\mathcal{M}^1 T_{,2} - \ell\mathcal{E}^1 S_{,2} - \Big(\mathcal{M}^1 T + \ell\mathcal{E}^1 S\Big)_{,2} + 2a\ell\, S_{,222} + \mathcal{O}_2\,, \\ &\delta_0\mathcal{M}^1 = -\ell\mathcal{E}^1 T_{,2} - \mathcal{M}^1 S_{,2} - \Big(\ell\mathcal{E}^1 T + \mathcal{M}^1 S\Big)_{,2} + 2a\ell\, T_{,222} + \mathcal{O}_2\,. \end{aligned} \tag{6.7}$$

The above result implies

$$\delta_0'\Gamma'' = \Gamma''' + C''' \,, \tag{6.8}$$

where the parameters T''', S''' are determined by the relations

$$\begin{aligned}
T''' &= T'S''_{,2} - T''S'_{,2} + S'T''_{,2} - S''T'_{,2} \,, \\
S''' &= S'S''_{,2} - S''S'_{,2} + T'T''_{,2} - T''T'_{,2} \,,
\end{aligned} \tag{6.9}$$

and $C''' \equiv C[T', S'\,;\, T'', S'']$ is the *central term* of the canonical algebra:

$$C''' = 2a\ell \int d\varphi \left(S''_{,2} T'_{,22} - S'_{,2} T''_{,22} \right) . \tag{6.10}$$

Combining Eqs. (6.6) and (6.8), one finds that the PB of the asymptotic symmetry generators has the form $\{\tilde{G}'', \tilde{G}'\} \approx \Gamma''' + C'''$, which implies the *weak* equality

$$\left\{ \tilde{G}'', \tilde{G}' \right\} \approx \tilde{G}''' + C''' \,. \tag{6.11}$$

It is known that the PB of two well defined generators must also be a well defined generator [25]. To promote (6.11) to the strong equality, we have to prove that there are no well defined asymptotic generators that weakly vanish (on the space of all solutions). Indeed, it has been shown in Appendix C that every asymptotic generator $\tilde{G}$ has a non-trivial surface term. Therefore, there holds the strong equality

$$\left\{ \tilde{G}'', \tilde{G}' \right\} = \tilde{G}''' + C''' \,. \tag{6.12}$$

C. Conservation laws

Let us first note that the improved total Hamiltonian $\tilde{H}_T$ is one of the generators $\tilde{G}[T,\, S]$. Indeed, the choice $T = 1$, $S = 0$ in (6.5) gives

$$\tilde{G}[1, 0] = -\ell\, \tilde{H}_T \,. \tag{6.13}$$

As a consequence, the commutator of the generator $\tilde{G}$ with the improved Hamiltonian $\tilde{H}_T$ *does not contain central term*:

$$\left\{ \tilde{G}[T, S],\, \tilde{H}_T \right\} = -\frac{1}{\ell} \left\{ \tilde{G}[T, S],\, \tilde{G}[1, 0] \right\} = -\frac{1}{\ell}\, \tilde{G}[\, S_{,2}\,, T_{,2}] \,. \tag{6.14}$$

The last term in the above equation is obtained by observing that $(T', S') = (1, 0)$, $(T'', S'') = (T, S)$ implies $C''' = 0$, $(T''', S''') = (S_{,2}\,, T_{,2})$. Now, with the help of relations (4.6), we find

$$\begin{aligned}
\frac{d}{dt}\tilde{G}[T, S] &= \frac{\partial \tilde{G}}{\partial t} + \left\{ \tilde{G},\, \tilde{H}_T \right\} \approx \frac{\partial \Gamma}{\partial t} - \frac{1}{\ell}\, \Gamma[S_{,2}\,,\, T_{,2}] \\
&= -\int d\varphi \left(\mathcal{M}^1 S_{,0} + \ell\, \mathcal{E}^1 T_{,0} \right) + \frac{1}{\ell} \int d\varphi \left(\mathcal{M}^1 T_{,2} + \ell\, \mathcal{E}^1 S_{,2} \right) = 0 \,.
\end{aligned}$$

Thus, the asymptotic generator $\tilde{G}[T,S]$ is conserved for every allowed choice of the parameters T, S. This also implies the conservation of the boundary term Γ, as $\tilde{G} \approx \Gamma$, and the constraints of the theory are conserved anyway:

$$\frac{d}{dt}\,\Gamma[T,S] \approx 0\,. \tag{6.15}$$

To test the obtained result, we shall calculate the value of all the conserved charges for the BTZ black hole solution (3.5), (3.6).

First, note that the black hole solution depends on the radial coordinate only, and consequently, the terms $\mathcal{E}^1$ and $\mathcal{M}^1$ in the surface integral Γ behave as constants. Second, the parameters T, S are periodic functions, as given in (4.7). This means that only *zero modes* in the Fourier decomposition of T, S survive the integration in Γ. Thus, there are only two independent non-vanishing charges for the black hole solution, and they are given by two inequivalent choices of *constants* T and S. If we take, let us say, $T=1$, $S=0$ as the first choice, and $T=0$, $S=1$ as the second one, all the other non-zero charges will be given as linear combinations of these two.

The particular choice $T=1$, $S=0$ gives $\Gamma[1,0]=-\ell E_0$, which means that the corresponding conserved charge is the energy E_0. Its value for the black hole solution is found to be $E_0 = 4\pi a m$, but taking into account that $a = 1/16\pi G = 1/4\pi$ (in units $4G=1$), we obtain

$$E_0(\text{black hole}) = m\,.$$

The second choice $T=0$, $S=1$, on the other hand, leads to $\Gamma[0,1]=-M_0$. The corresponding conserved charge is the angular momentum M_0, and its black hole value reads

$$M_0(\text{black hole}) = J\,.$$

We see that constants m and J, which parametrize the black hole solution, have the meaning of energy and angular momentum, respectively. We also see that there are no other independent conserved charges. Geometrically, these two charges parametrize globally inequivalent asymptotically AdS spaces [22].

D. Central charge

The canonical algebra (6.12) can be brought to a more recognizable form by using the representation in terms of Fourier modes. The solutions (4.7) for the parameters T and S are then written in the form:

$$\begin{aligned} T &= \sum_{-\infty}^{\infty}\left(a_n e^{inx^+} + \bar{a}_n e^{inx^-}\right)\,, \\ S &= \sum_{-\infty}^{\infty}\left(a_n e^{inx^+} - \bar{a}_n e^{inx^-}\right)\,, \end{aligned} \tag{6.16}$$

where $x^\pm = (t/\ell) \pm \varphi$, and the reality of T,S implies $a_{-n} = a_n^*$, $\bar{a}_{-n} = \bar{a}_n^*$. The asymptotic generator $\tilde{G}[T,S]$ is a linear, homogeneous function of its parameters T and S, since the

asymptotic symmetry is defined up to the residual gauge transformations generated by $\tilde{G}[T = S = 0]$. Therefore, using the above Fourier decomposition, we can write

$$\tilde{G}[T, S] = -2 \sum_{-\infty}^{\infty} \left(a_n L_n + \bar{a}_n \bar{L}_n \right) . \tag{6.17}$$

The new asymptotic generators L_n, $\bar{L}_n$ are also defined up to residual gauge terms, and the reality of $\tilde{G}[T, S]$ is expressed by demanding $L_{-n} = L_n^*$, $\bar{L}_{-n} = \bar{L}_n^*$. Solving equation (6.17) in terms of L_n, $\bar{L}_n$, one finds:

$$\begin{aligned} 2L_n &= -\tilde{G}[T = S = e^{inx^+}] \\ 2\bar{L}_n &= -\tilde{G}[T = -S = e^{inx^-}] \, . \end{aligned} \tag{6.18}$$

Now, we can rewrite the canonical PB algebra (6.12) in terms of the new asymptotic generators. The definitions (6.18) and the relations (6.9), (6.10) lead to

$$\{L_n, \, L_m\} = -i(n - m)L_{n+m} - 2\pi i \, a\ell \, n^3 \delta_{n,-m} \, , \tag{6.19a}$$

$$\left\{\bar{L}_n, \, \bar{L}_m\right\} = -i(n - m)\bar{L}_{n+m} - 2\pi i \, a\ell \, n^3 \delta_{n,-m} \, . \tag{6.19b}$$

$$\left\{L_n, \, \bar{L}_m\right\} = 0 \, , \tag{6.19c}$$

Upon the redefinition of the zero modes, $L_0 \to L_0 + \pi a\ell$, $\bar{L}_0 \to \bar{L}_0 + \pi a\ell$, we obtain the standard form of the Virasoro algebra with the *classical* central charge:

$$\{L_n, \, L_m\} = -i(n - m)L_{n+m} - 2\pi i \, a\ell \, n(n^2 - 1)\delta_{n,-m} \, , \tag{6.20a}$$

$$\left\{\bar{L}_n, \, \bar{L}_m\right\} = -i(n - m)\bar{L}_{n+m} - 2\pi i \, a\ell \, n(n^2 - 1)\delta_{n,-m} \, . \tag{6.20b}$$

Using the standard string theory normalization of the central charge, we have

$$c = 12 \cdot 2\pi a\ell = \frac{3\ell}{2G} \, . \tag{6.21}$$

Thus, the value of the central charge in the theory of gravity with torsion coincides with the Brown–Henneaux central charge of Einstein's theory with cosmological constant, defined in Riemannian spacetime.

The form (6.20) of the asymptotic algebra shows that central term for the AdS subgroup, generated by the generators (L_{-1}, L_0, L_1), $(\bar{L}_{-1}, \bar{L}_0, \bar{L}_1)$, vanishes. This is a consequence of the fact that AdS subgroup is an exact symmetry of the vacuum (3.2), (3.3), in agreement with the result of the first reference in [1]. The latter states that non-trivial classical central term does not exist if at least one exact solution is left invariant under the action of the asymptotic symmetry group.

VII. CONCLUDING REMARKS

We presented here an investigation of the structure of asymptotic symmetry in 3d gravity with torsion. We have chosen a specific form of the Baekler–Mielke action [13,14] which

yields the teleparallel dynamics, in order to isolate the influence of torsion on the asymptotic dynamics.

Our procedure for constructing vacuum solutions is based on the maximally symmetric form of the Riemannian piece of the curvature in Riemann–Cartan geometry, equation (2.5). We obtained two exact vacuum solutions, the AdS solution and the black hole, both in the realm of the teleparallel geometry.

The results concerning the asymptotic symmetry of the theory are based on a natural definition of asymptotically AdS behaviour of dynamical variables. Canonical analysis of the related generators reveals the necessity for an improvement of their form by the addition of appropriate boundary terms, which are interpreted as the conserved charges of the teleparallel theory. The canonical algebra of the generators is realized as the Virasoro algebra with the classical central charge $c = 3\ell/2G$. The fact that the value of this charge is the same as in Riemannian spacetime of general relativity indicates that the boundary dynamics in 3d gravity depends much more on the form of asymptotic conditions than on the underlying geometry.

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APPENDIX A: SYMMETRIES OF THE ADS VACUUM

The invariance condition for the AdS vacuum (3.2), (3.3) leads to the following relations:

$$
\begin{aligned}
&\xi^0{}_{,0} = -\frac{r}{\ell^2 f^2}\,\xi^1 \qquad \xi^1{}_{,1} = \frac{r}{\ell^2 f^2}\,\xi^1 \qquad \xi^2{}_{,2} = -\frac{1}{r}\,\xi^1 \\
&\xi^1{}_{,0} = f^4 \xi^0{}_{,1} \qquad \xi^0{}_{,2} = \frac{r^2}{f^2}\,\xi^2{}_{,0} \qquad \xi^1{}_{,2} = -r^2 f^2 \xi^2{}_{,1} \\
&\theta^0 = -r f \xi^2{}_{,1} \qquad \theta^1 = \frac{f}{r}\,\xi^0{}_{,2} \qquad \theta^2 = -\frac{1}{f^2}\,\xi^1{}_{,0}
\end{aligned}
$$

The general solution of these equations has the form

$$
\begin{aligned}
&\xi^0 = \ell\sigma_1 - \frac{r}{f}\partial_2 Q\,, \qquad \xi^1 = \ell^2 f \partial_0\partial_2 Q\,, \qquad \xi^2 = \sigma_2 - \frac{\ell^2 f}{r}\partial_0 Q\,, \\
&\theta^0 = -\frac{\ell^2}{r}\,\partial_0 Q\,, \qquad \theta^1 = Q\,, \qquad \theta^2 = \frac{1}{f}\,\partial_2 Q\,,
\end{aligned} \tag{A1}
$$

where

$$
Q \equiv \sigma_3 \cos x^+ + \sigma_4 \sin x^+ + \sigma_5 \cos x^- + \sigma_6 \sin x^-\,, \tag{A2}
$$

and σ_i are six arbitrary dimensionless parameters. For the basis of six independent Killing vectors we can take:

$$
\begin{aligned}
\xi_{(1)} &= (\ell\,,\,0\,,\,0)\,, \\
\xi_{(2)} &= (0\,,\,0\,,\,1)\,, \\
\xi_{(3)} &= \left(\frac{r}{f}\,\sin x^+\,,\;-\ell f\cos x^+\,,\;\frac{\ell f}{r}\sin x^+\right) \\
\xi_{(4)} &= \left(\frac{r}{f}\,\cos x^+\,,\;\ell f\sin x^+\,,\;\frac{\ell f}{r}\cos x^+\right) \\
\xi_{(5)} &= \left(\frac{r}{f}\,\sin x^-\,,\;\ell f\cos x^-\,,\;-\frac{\ell f}{r}\sin x^-\right) \\
\xi_{(6)} &= \left(\frac{r}{f}\,\cos x^-\,,\;\ell f\sin x^-\,,\;-\frac{\ell f}{r}\cos x^-\right)\,.
\end{aligned}
\tag{A3}
$$

We have explicitly verified that the above expressions for ξ^μ and θ^i fall into the class of asymptotic transformations (4.5). In particular, six inequivalent solutions for the parameters T and S define the six Killing vectors (A3) of the teleparallel AdS vacuum solution.

APPENDIX B: ASYMPTOTIC FORM OF THE CONSTRAINTS

Using the adopted asymptotic conditions (4.2), the secondary constraints

$$
R^i{}_{\alpha\beta} \approx 0\,, \qquad T^i{}_{\alpha\beta} - \frac{2}{\ell}\,\varepsilon^i{}_{jk}\,b^j{}_\alpha b^k{}_\beta \approx 0
$$

are rewritten in the form

$$
\omega^0{}_1 = \mathcal{O}_4\,, \qquad \omega^2{}_1 = \mathcal{O}_4\,, \tag{B1a}
$$

$$
(r\alpha_2)_{,1} = \mathcal{O}_3\,, \qquad (r\beta_2)_{,1} = \mathcal{O}_3\,, \tag{B1b}
$$

$$
\ell\,(\Omega^2{}_2 - \Omega^0{}_2) + r^2\Omega^1{}_1 = [r\,(B^2{}_2 - B^0{}_2)]_{,1} + \mathcal{O}_3\,, \tag{B1c}
$$

$$
B^0{}_2 + B^2{}_2 + \tfrac{r^2}{\ell}\,B^1{}_1 - \ell\,\Omega^0{}_2 = (rB^2{}_2)_{,1} + \mathcal{O}_3\,, \tag{B1d}
$$

where

$$
\alpha_\mu \equiv \omega^0{}_\mu + \frac{1}{\ell}\,b^2{}_\mu - \frac{1}{\ell}\,b^0{}_\mu\,, \qquad \beta_\mu \equiv \omega^2{}_\mu + \frac{1}{\ell}\,b^0{}_\mu - \frac{1}{\ell}\,b^2{}_\mu\,. \tag{B2}
$$

From (B1b), we see that the terms $\mathcal{E}^\alpha$ and $\mathcal{M}^\alpha$, included in the surface integrals, have the asymptotic behaviour

$$
\mathcal{E}^1{}_{,1} = \mathcal{O}_3\,, \quad \mathcal{E}^2 = \mathcal{O}_3\,, \qquad \mathcal{M}^1{}_{,1} = \mathcal{O}_3\,, \quad \mathcal{M}^2 = \mathcal{O}_3\,,
$$

wherefrom one verifies the finiteness of the surface term Γ for every region of integration (not necessarily a circle).

Beside the constraints, the equations of motion also refine the asymptotic behaviour of the fields. This is because we demand that the adopted asymptotics is conserved in time, which imposes additional restrictions at infinity. Thus, one finds

$$
\ell\,\alpha_0 + \beta_2 = \mathcal{O}_3\,, \qquad \ell\,\beta_0 + \alpha_2 = \mathcal{O}_3\,, \tag{B3a}
$$

$$
\ell\,\Omega^0{}_0 + \Omega^0{}_2 = \mathcal{O}_3\,, \qquad \ell\,\Omega^2{}_0 + \Omega^2{}_2 = \mathcal{O}_3\,, \tag{B3b}
$$

$$
B^0{}_0 + B^2{}_0 + \tfrac{r^2}{\ell^2}\,B^1{}_1 - \ell\,\Omega^2{}_0 = (rB^0{}_0)_{,1} + \mathcal{O}_3\,. \tag{B3c}
$$

The terms $\Omega^i{}_\mu$ and $B^i{}_\mu$ are defined in (4.2).

APPENDIX C: NON-TRIVIALITY OF THE BOUNDARY TERMS

In this Appendix, we shall examine if some of the generators $\tilde{G}$ have trivial central terms, or equivalently, if the boundary term $\Gamma[T,S]$ vanishes on-shell for some values of the parameters T and S.

First, note that all physically acceptable solutions of the equations of motion are gauge equivalent to the black hole solution (3.5), (3.6) [21]. As our asymptotic conditions restrict the full gauge group to the subgroup (4.5), we shall consider the set of solutions $\mathcal{W}$, defined by

$$\begin{aligned} \mathcal{W} \;=\; & \text{the black hole solution, plus all its transforms} \\ & \text{under the action of the gauge subgroup (4.5).} \end{aligned}$$

Thus, solving the equation $\Gamma'' = 0$ in terms of $(T'',\ S'')$ on the space of solutions $\mathcal{W}$ is equivalent to solving the equation

$$\bar{\Gamma}'' + \delta_0'\bar{\Gamma}'' = 0 \quad \text{for all } (m,J) \text{ and } (T',S')\,. \tag{C1}$$

The bar over Γ denotes that the boundary term is evaluated on the two-parameter space of the black hole solutions (3.5), (3.6).

For $T' = S' = 0$, we find the condition $\bar{\Gamma}'' = 0$, or equivalently,

$$\int T'' d\varphi = \int S'' d\varphi = 0\,. \tag{C2}$$

Thus, the zero modes of the functions T'', S'' must vanish. Next, using (6.8), equation (C1) can be brought to $\bar{\Gamma}''' + C''' = 0$ for all (m,J) and (T',S'), which means that

$$\int T''' d\varphi = \int S''' d\varphi = 0 = C''' \quad \text{for all } (T',S')\,.$$

We now concentrate on the first two equations rewritten as

$$\int d\varphi \left(S'T''_{,2} + T'S''_{,2} \right) = \int d\varphi \left(S'S''_{,2} + T'T''_{,2} \right) = 0\,.$$

By adding and subtracting the two equations, they take the form

$$\int d\varphi\,(T'+S')(T''+S'')_{,2} = \int d\varphi\,(T'-S')(T''-S'')_{,2} = 0$$

for all allowed values of T' and S'. Substituting here the general solution (4.7) for the parameters T and S, one finds

$$\int d\varphi\, f' f''_{,2} = \int d\varphi\, g' g''_{,2} = 0 \quad \forall\ f', g',$$

which can hold only if f'' and g'' are constants. Consequently,

$$S'' = \text{ const}, \quad T'' = \text{ const}. \tag{C3}$$

Taking into account the condition (C2), this result implies

$$\Gamma = 0 \quad \Leftrightarrow \quad S = T = 0\,. \tag{C4}$$

Therefore, *only the trivial asymptotic generator* $\tilde{G}[0,0]$ *has the vanishing surface term.* We can now be sure that the algebra (6.12) is valid *off-shell*, because there are no well defined asymptotic generators that weakly vanish [25].

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